

The moduli space of hyper-Kähler four-fold compactifications

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ABSTRACT: I discuss some aspects of the moduli space of hyper-Kähler four-fold compactifications of type II and \mathcal{M} - theories. The dimension of the moduli space of these theories is strictly bounded from above. As an example I study $\text{Hilb}^2(K3)$ and the generalized Kummer variety $K^2(T^4)$. In both cases RR-flux (or G -flux in \mathcal{M} -theory) must be turned on, and we show that they give rise to vacua with $\mathcal{N} = 2$ or $\mathcal{N} = 3$ supersymmetry upon turning on appropriate fluxes. An interesting subtlety involving the symmetric product limit $S^2(K3)$ is pointed out.

KEYWORDS: Flux compactifications, Superstring Vacua.

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1. Introduction

Compactifications of type II strings on hyper-Kähler two-folds has been much studied and very well understood (see the review [1] and references therein for an excellent overview). The reason for this happy state of affairs is that any two compact hyper-Kähler two-folds are diffeomorphic to each other and there is essentially only one $K3$ surface. The moduli space of $K3$ surfaces can be determined precisely, and in string theory we see this simplicity as the fact that type IIA on a $K3$ surface is dual to heterotic strings on T^4 whose moduli space is the Narain moduli space $Gr(4, 20)$. An analogous understanding of hyper-Kähler four-folds is lacking in literature. In fact there are still only two known examples of compact hyper-Kähler four-folds even though the cohomology of a compact hyper-Kähler four-fold has been understood for a long time. Any treatment of compactifications on hyper-Kähler four-folds suffers from the fact that there are so few examples. Fortunately, it turns out

that one can map out the moduli space of type II string theories on compact hyper-Kähler four-folds using simple CFT arguments [2]. We review the argument that obtains the moduli space of $\mathcal{N} = (4, 4)$ SCFTs in appendix 1 and use the results of [2] to obtain the moduli space of type IIA /B compactifications on hyper-Kähler four-folds. There is an action of $O(4, b_2 - 2; \mathbf{Z})$ on the moduli space of hyper-Kähler four-fold compactifications of type IIA, which was observed by Verbitsky as the group acting on the lattice $H^*(X; \mathbf{Z})$ for an arbitrary hyper-Kähler manifold. We point out that there is a simple reason why this group acts on $H^*(X; \mathbf{Z})$. Somewhat surprisingly, it is possible to show that these theories have a moduli space of bounded dimension, essentially due to the fact that the topological types of hyper-Kähler four-folds is bounded. It would be interesting to obtain a simple physical understanding of this fact.

In section 2 the basic facts of hyper-Kähler four-folds is summarized. Sections 3 and 4 work out aspects of the dimensional reduction of type II theories on hyper-Kähler four-folds. In most respects this is similar to the case of Calabi-Yau four-folds and we follow the paper of Gates, Gukov and Witten [3] in performing this reduction.

Some of the type IIA/ \mathcal{M} -theory compactifications will be not be solutions of the 1-loop effective action coming from string theory. We analyze this in more detail in section 5. In section 6 we work out the conditions under which the two known examples of hyper-Kähler four-folds yield supersymmetric vacua. We also show that the symmetric product $S^2(K3)$ does not arise in the moduli space of hyper-Kähler compactifications with fluxes.

Though we are only talking about hyper-Kähler four-folds in this paper, it appears that there are few other ways of obtaining $\mathcal{N} = 3$ supersymmetric theories in 3d. In particular there is no other class of compactifications that yield $\mathcal{N} = 3$ supersymmetry in 3d at weak coupling. So one may view our results as indicating that the moduli space of $\mathcal{N} = 3$ theories in 3d (and the moduli space of $\mathcal{N} = (3, 3)$ supergravity theories in 2d) are very tightly constrained. In particular the moduli space is of finite dimension with a strict upper bound on the dimension. It has already been noted that a similar thing happens for Calabi-Yau compactifications in general, in that the dimension of the moduli space of $\mathcal{N} = 2$ supergravity theories in 4d for example is expected to be finite. What is perhaps surprising is the simplicity of showing this for $\mathcal{N} = 3$ supergravities in 3d. In particular, suppose we consider $\mathcal{N} = 4$ supergravity in 3d, we know that a class of these theories arise via compactification on $K3 \times K3$ whose moduli space is of finite dimension. However these compactifications do not exhaust all $\mathcal{N} = 4$ supergravity theories in 3d and in particular a large class of such compactifications arise via dimensional reduction on $T^2 \times CY_3$ for which there are only indirect arguments that suggest a bound on the dimension of the moduli space.

Flux compactification in \mathcal{M} -theory and type II context has a long history (see [4] for overview and references). However, the recent work of Aspinwall and Kallosh [5] on $K3 \times K3$ is very closely related to the analysis in this paper, and some of the techniques used there are applied to the case of hyper-Kähler four-folds here.

2. Some facts on Hyper-Kähler four-folds

A hyper-Kähler 4-fold is a Kähler manifold with a nowhere vanishing non degenerate holomorphic 2-form ω . Then ω^2 trivializes the canonical line bundle, so by Yau's proof of Calabi conjecture, there is a unique Ricci-flat metric that respects the hyper-Kähler structure. The cohomology of a general Kähler manifold can be decomposed via Hodge decomposition. For a hyper-Kähler 4-fold, the non trivial Hodge numbers are $h^{1,1}, h^{2,1}, h^{3,1}$ and $h^{2,2}$. However, not all of them are independent. Given any type $(1,1)$ -form we can create a $(3,1)$ form by wedging with ω , so that $h^{3,1} = h^{1,1}$. Also $h^{1,1} \geq 1$ as the space is Kähler, so we can write $h^{1,1} = 1+p$ for some p in \mathbf{Z}^+ . Furthermore, just as for a Calabi-Yau 4-fold, $h^{2,2}$ is not independent. The quickest way to note this is to consider the index of the Dolbeault operator $\bar{\partial}_{E_2}$ acting on the bundle E_2 of holomorphic type $(2,0)$ forms. This index is given by:

$$\text{Ind} \bar{\partial}_{E_2} = \sum_{q=0} (-1)^q h^{2,q} \tag{2.1}$$

However, the index also has a purely topological character, and can be expressed via the Atiyah-Singer Index theorem as:

$$\text{Ind}(\bar{\partial}_{E_2}) = \int \text{Todd}(X) \text{Ch}(\Omega^{2,0}) \tag{2.2}$$

Using the standard expression for the Todd genus and Chern character, we compute:

$$\text{Ind}(\bar{\partial}_{E_2}) = \frac{1}{120} \int (3c_2^2 + 79c_4) \tag{2.3}$$

where we used the fact that $c_1 = 0$. Now, the Todd genus of a hyper-Kähler 4-fold is precisely 3, and this implies a relation between c_2^2 and c_4 (incidentally, $\int c_4$ is the arithmetic genus or Euler characteristic of the hyper-Kähler 4-fold X). Specifically:

$$\text{Todd}(X) = \frac{1}{720} (3c_2^2 - c_4) \tag{2.4}$$

so that

$$\int_X c_2^2 = 720 + \frac{\chi}{3} \tag{2.5}$$

Using (2.5) in (2.3) we get a relation between the various Hodge numbers. Denoting $h^{2,1} = 2q^1$ this relation is:

$$h^{2,2} = 72 + 8p - 4q \tag{2.6}$$

So the hyper-Kähler 4-folds are characterized by two non negative integers (p, q) .

¹Here we used the fact that b_3 is divisible by 4, for a hyper-Kähler four-fold. Incidentally, this also implies χ is divisible by 12, which is a stronger result than the one for Calabi-Yau four-folds. The Hilbert scheme of two points on K3 gives us an example where χ is divisible by 12, and not by 24, so this is the strongest result we can get. In our notation $\frac{\chi}{24} = \frac{1}{2}(7 + p - q)$.

3. Compactification of type IIA on Hyper-Kähler four-folds

In this section we will describe the compactification of type IIA string theory on a hyper-Kähler four-fold X . In the large volume limit these compactifications can be discussed by dimensionally reducing type IIA supergravity on hyper-Kähler four-folds.

The bosonic content of type IIA supergravity in ten dimensions is the metric g_{MN} , an antisymmetric two-form B_{MN} and dilaton ϕ from the NS-NS sector. The R-R sector gives rise to the one-form gauge field A_M and three form C_{MNP} . The bosonic action in string frame is of the form:

$$L = \int d^{10}x \sqrt{-g} [e^{-2\phi} (R^{10} + 4(\nabla\phi)^2 - \frac{1}{12}H^2) - \frac{1}{4}F^2 - \frac{1}{48}G^2] \quad (3.1)$$

Where:

$$F = dA \quad H = dB \quad G = dC + A \wedge H \quad (3.2)$$

are the gauge invariant field strengths. The action (3.1) is of course the tree level action for type IIA string theory in ten dimensions. There are higher order terms in the effective action that are not captured in (3.1). For the most part their structure is not known. There is however an important term of the form $B \wedge X_8$ where X_8 is a particular contraction of four powers of the Riemann tensor. This term was shown to be present in type IIA by considering scattering amplitudes in type II string theory [6]. This term leads to a tadpole for the B -field which has to be cancelled in type IIA by turning on G -flux and/ or adding N F1-strings such that:

$$N = \frac{\chi}{24} - \frac{1}{2(2\pi)^2} \int_X G \wedge G \quad (3.3)$$

If the Euler number of X is not divisible by 24, then the tadpole cannot be canceled by simply adding F-strings and we must turn on RR-flux G also. Of course, turning on G -flux we will typically end up breaking supersymmetry unless the G -flux happens to be primitive with respect to the \mathbf{P}^1 of complex structures on X . For the moment we will ignore these subtleties and address them in section 3. The action for type IIA supergravity is invariant with respect to 32 supercharges, 16 of which are left-chiral and 16 right-chiral with respect to the chirality operator in 10d. Upon compactifying on X , the resulting action in two dimensions possesses residual supersymmetry only if X admits a covariantly constant spinor. In the case of hyper-Kähler four-folds the holonomy group of X is $sp(2)$. A generic eight dimensional spinor is in one of the two inequivalent spinor representations of $spin(8)$ say $\mathbf{8}_+$. Under $sp(2)$ we have the decomposition:

$$\mathbf{8}_+ = \mathbf{5} + \mathbf{1} + \mathbf{1} + \mathbf{1} \quad \mathbf{8}_- = \mathbf{4} + \mathbf{4} \quad (3.4)$$

so that there is a three-dimensional space of covariantly constant spinors on X . Via the decomposition:

$$\mathbf{16} = (\mathbf{8}_+, +) + (\mathbf{8}_-, -), \quad \mathbf{16}' = (\mathbf{8}_+, -) + (\mathbf{8}_-, +), \quad (3.5)$$

corresponding to $SO(1, 9) \rightarrow SO(8) \times SO(1, 1)$ we end up with a non-chiral two dimensional supergravity theory² with $\mathcal{N} = (3, 3)$ supersymmetry upon compactifying type IIA on X .

To determine the spectrum of the resulting two dimensional theory one performs Kaluza-Klein reduction of the various fields of type IIA. As the resulting two dimensional theory is non-chiral the fermions simply arise as $\mathcal{N} = (3, 3)$ superpartners and it is enough to count the massless bosonic degrees. These are associated to the harmonics of the various bosonic fields of type IIA. Denoting the holomorphic 2-form on X by ω , one can expand the B_{MN} zero modes as:

$$B = \sum_i b^i \omega_i^{1,1} + b\omega \tag{3.6}$$

where:

$$\omega^{1,1} \in H^{1,1}(X) \quad b \in \mathbf{C} \quad b^i \in \mathbf{R} \tag{3.7}$$

leading to $h^{1,1} + 2$ scalars. The C_{MNP} zero modes lead to $2h^{2,1}$ scalars and $h^{1,1} + 2$ vectors via:

$$C = \sum_j c^j \omega_j^{2,1} + \sum_n C_\mu^n \omega_n^{1,1} + C_\mu \omega \quad \omega^{2,1} \in H^{2,1}(X) \quad c^j \in \mathbf{C} \tag{3.8}$$

The metric deformations lead to $3h^{1,1} - 2$ scalars g^k as follows: The zero modes of the graviton satisfy the Lichnerowicz equation which in a suitable gauge can be written as:

$$D_k D^k h_{ij} - R_{isjt} h^{st} = 0 \tag{3.9}$$

It is easy to see that the metric variations of the form δh_{ab} and $\delta h_{a\bar{b}}$ do not mix in (3.9) so they can be considered separately. For every element $\omega^{1,1}$ one obtains a variation of the form $\delta h_{a\bar{b}}$ so that the number of such deformations is $h^{1,1}$. Similarly, given $\omega^{1,1} \in H^{1,1}(X)$ one can construct a variation of type δh_{ab} as:

$$\delta h_{ab} = \omega_{(a}^{\bar{c}} \omega_{b)}^{1,1} \tag{3.10}$$

However if $\omega^{1,1}$ is proportional to the Kähler form then (3.10) vanishes, so that there are only $2h^{1,1} - 2$ deformations of type δh_{ab} so that the space of $sp(2)$ holonomy metrics on a hyper-Kähler four-fold has dimension $3h^{1,1} - 2$.

Collecting all the matter content together we end up with $h^{1,1} = (p + 1)$ $\mathcal{N} = (4, 4)$ vector multiplets containing g^k, b_i as the scalar components, together with q $\mathcal{N} = (4, 4)$ hyper multiplets containing the $4q$ scalars c^j . Even though we have only $\mathcal{N} = (3, 3)$ supersymmetry, the matter sector arranges itself into $\mathcal{N} = (4, 4)$ multiplets, which is a familiar fact given that any supersymmetric sigma model with $\mathcal{N} = 3$ supersymmetry is automatically $\mathcal{N} = 4$ supersymmetric also. Of course the higher order terms in the effective action will only be $\mathcal{N} = (3, 3)$ supersymmetric.

The supergravity sector contains the graviton, three abelian gauge fields and a scalar, along with three gravitini and three Majorana fermions. The dilaton sits in the supergravity multiplet.

²In (3.5) the $\mathbf{16}$ and $\mathbf{16}'$ refer to the ten dimensional Majorana-Weyl spinors of opposite chirality associated to type IIA, whereas the spinor representations of $SO(1, 1)$ are labeled by their charges under $spin(1, 1)$.

The low energy effective action for the vector and hyper-multiplet moduli will in general be given by a $\mathcal{N} = (4, 4)$ supersymmetric sigma model. In the case of the vector multiplets with rigid supersymmetry this sigma model is based on a target space that is hyper-Kähler with torsion (HKT), so we expect upon coupling to supergravity that the target space is quaternionic Kähler with torsion (QKT). The hyper multiplet moduli space is similarly a hyper-Kähler or Quaternionic Kähler manifold. As the two multiplets carry scalars with different R-symmetries the moduli space factorizes just as in $\mathcal{N} = 2$ supergravity coupled to matter in four dimensions. Denoting the moduli space \mathcal{M} of type IIA on a hyper-Kähler four-fold as:

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H \tag{3.11}$$

what can be said about \mathcal{M}_V and \mathcal{M}_H ?

The worldsheet description of any $\mathcal{N} = (3, 3)$ supersymmetric compactification to two dimensions is in the form of a $\mathcal{N} = 4$ SCFT with small $\mathcal{N} = 4$ SCA and $c = 12$. The space-time moduli that sit in the $(p + 1)$ vector multiplets are all $\mathcal{N} = 4$ chiral primary operators of this internal $\mathcal{N} = 4$ SCA. Since any $\mathcal{N} = 4$ SCA has a $SU(2)_L \times SU(2)_R$ R-symmetry this implies³ that the moduli space \mathcal{M}_V has a $SO(4)$ isometry. It turns out due to a theorem of Berger and Simons (see [1] for a nice discussion on the Berger-Simons result) that the smooth manifolds with $SO(4)$ holonomy and dimension greater than 4 are only the symmetric spaces, the so called Grassmann manifolds. This leads us to identify:

$$\mathcal{M}_V = \frac{O(4, p + 1)}{O(4) \times O(p + 1)} \tag{3.12}$$

There is a natural $O(4, p + 1; \mathbf{Z})$ symmetry of the moduli space which we can quotient by maintaining the Hausdorff property of the the resulting space. It is natural to conjecture that the U-duality group for this theory is $O(4, p + 1; \mathbf{Z})$.

In type IIA the dilaton ϕ is in the $\mathcal{N} = (3, 3)$ supergravity multiplet. This implies that the form of the moduli space is completely independent of string coupling $g_s = e^\phi$. For large g_s , type IIA goes over to 11d supergravity which is the low energy limit of \mathcal{M} -theory. This means the \mathcal{M} -theory moduli space is also given by (3.12). By the same argument, the metric on the moduli space is independent of string coupling.

Given the moduli space of the form (3.12), we can take the large radius limit. The large radius limit can be determined by examining the Dynkin diagram of $O(4, p + 1)$, and it turns out that the structure of the moduli space in the large radius limit is given by:

$$\mathcal{M} = \frac{O(3, p)}{O(3) \times O(p)} \times \mathbf{R}_+ \times \mathbf{R}^{p+3} \tag{3.13}$$

This is what we expect in the large radius limit. In this limit we expect the metric deformations to be characterized by the moduli space of $sp(2)$ holonomy metrics of fixed volume of a hyper-Kähler four-fold, which is the $O(3, p)$ factor, the \mathbf{R}_+ factor corresponds to the trivial radial mode. The \mathbf{R}^{p+3} factor corresponds to the scalars arising from dimensional reduction of the NSNS 2-form. This provides a non-trivial consistency check.

³Details of this standard argument are provided in appendix 1. This argument was first applied for determining the moduli space of $\mathcal{N} = 4$ SCFTs by Cecotti.

The $\mathcal{N} = (3, 3)$ supergravity coupled to matter has not been constructed in literature. There is however the case of $\mathcal{N} = (4, 4)$ supergravity coupled to matter which has been analysed [7]. This theory has a gauged $SU(2) \in SO(4)$ R-symmetry and it has been shown that the target space parameterized by the scalars in this theory can be hyper-Kähler or Quaternionic Kähler. We expect a similar result to hold even in the case of $\mathcal{N} = (3, 3)$ supergravity coupled to matter. That is, with $\mathcal{N} = (3, 3)$ supersymmetry, the form of the moduli space remains non-trivial in general. This raises the puzzle as to how the CFT analysis was able to determine the local form of the moduli space as (3.12). We will resolve this puzzle in the next section.

One subtlety that has to be pointed out is that there is a difference between the $K3$ case and the case of general hyper-Kähler manifolds which affects our understanding of the moduli space. For $K3$ surfaces the global Torelli theorem holds, so that the moduli space of complex structures is determined by the space of periods. It is the space of periods that the supergravity analysis is sensitive to, and so is the chiral primary ring of the $\mathcal{N} = (4, 4)$ worldsheet theory. It is not known whether a version of the global Torelli theorem holds for the higher dimensional cases. If it does not, then the choice of periods does not determine the complex structure fully. What will be lacking is some discrete data. It is known that all hyper-Kähler manifolds are deformations of a projective variety so they all have $\pi_1 = 0$. So it is not possible to have discrete torsion [8] in the worldsheet SCFT. I do not know what extra data the SCFT can have in this case that is not captured by the chiral primary ring. So the analysis of the moduli space in this paper is carried out modulo the discrete ambiguity arising from lack of a global Torelli like theorem.

3.1 \mathcal{M} -theory on hyper-Kähler four-folds

The low energy limit of \mathcal{M} theory is 11d supergravity whose bosonic content is a graviton and a 3-form potential A , with four-form flux G .

Dimensional reduction of 11d supergravity on a hyper-Kähler four-fold yields a three dimensional $\mathcal{N} = 3$ supergravity coupled to matter. The matter multiplets are the vector multiplet (whose bosonic content is three scalars transforming as $\mathbf{3}$ of the $SO(3)$ R-symmetry together a gauge field) and the hyper multiplet (which contains four scalars transforming as a complex doublet of the R-symmetry). Any action for the hyper multiplets is automatically $\mathcal{N} = 4$ supersymmetric, so is the low energy effective action for the vector multiplets (in the absence of G-flux). Upon dimensional reduction, we end up with a $\mathcal{N} = 3$ supergravity multiplet with a graviton, three gravitini. The matter sector consists $p+1$ vector multiplets (after dualizing some vectors into scalars) and q hyper multiplets. The moduli space factorizes as in the type IIA case. Upon dualizing the vectors into scalars, we expect the \mathcal{M} -theory moduli space to coincide with the type IIA case. The \mathcal{M} -theory moduli space will be of the form:

$$\mathcal{M}_{11d} = \frac{O(4, p+1)}{O(4) \times O(p+1)} \otimes \mathcal{M}_H \tag{3.14}$$

4. Compactification of type IIB on hyper-Kähler four-folds

The compactification of type IIB string theory on a hyper-Kähler four-fold X leads to a two dimensional $\mathcal{N} = (0, 6)$ supersymmetry theory in the non-compact directions. Its low energy limit is $\mathcal{N} = (0, 6)$ supergravity coupled to matter. In this section we determine the matter content of this theory and the moduli space. In the large volume limit type IIB string theory in ten dimensions is well approximated by type IIB supergravity. The bosonic content of type IIB supergravity is the graviton g_{MN} , the anti-symmetric two form B_{MN} , the dilaton ϕ , the RR axion C , along with the RR two form A_{MN} and the self-dual four-form G_{MNPQ} . Type IIB in ten dimensions has a $sl_2(\mathbf{Z})$ action where the two forms A, B form a doublet of $sl_2(\mathbf{Z})$ and the axio-dilaton can be combined as:

$$\tau = c + ie^{-\phi} \tag{4.1}$$

and transforms under $sl_2(\mathbf{Z})$ as:

$$\tau \rightarrow \frac{(a\tau + b)}{(c\tau + d)} \quad a, b, c, d \in \mathbf{Z} \quad ad - bc = 1 \tag{4.2}$$

As the five form field strength:

$$F = dG + \frac{3}{4}B \wedge dB \tag{4.3}$$

is self dual, there is no covariant action whose equation of motion yields the self-duality constraint. Agreeing to impose this constraint by hand, we can write down a lagrangian for type IIB supergravity. As in the type IIA case, we need to determine the massless spectrum of particles in the 2d theory. The NS-NS sector modes g, B and ϕ give rise to the same zero modes for both type IIA and IIB. So we end up with $4h^{1,1} + 1$ scalars from the NS-NS sector.

In type IIA we argued that the dilaton went into the supergravity multiplet. In type IIB it is the fluctuation of the radial mode of the metric that goes into the $\mathcal{N} = (0, 6)$ supergravity multiplet.

From the RR sector, the axion gives rise to a real scalar. The RR two form A gives rise to $h^{1,1} + 2$ scalars exactly as the B-field. The expansion of the self-dual five-form F is more complicated. It can be expanded as follows:

$$F = \sum dC^i \omega^4_{-i} + \sum dC'^j \omega^4_{+j} \tag{4.4}$$

where ω^4_{-} refers to the space of anti self-dual four-forms on X , while ω^4_{+} refers to the space of self-dual four-forms. The self-duality of F implies that C^i are anti self-dual scalars, while C'^j are self-dual. This means the scalars C^i are left-moving while the scalars C'^j are right moving. In this notation the supersymmetries of the 2d theory are purely right-moving.

Therefore, in the purely right-moving matter sector we have $b^4_{-} + 5h^{1,1} + 3$ scalars.

The middle dimensional cohomology of X decomposes into self-dual and anti self-dual pieces by Poincare duality. The signature σ of X is nothing but $\sigma = b^4_{+} - b^4_{-}$. The Hirzebruch signature theorem relates σ to the Euler character of X :

$$\sigma = \frac{1}{45} \int (7p_2 - p_1^2) = 48 + \frac{\chi}{3} \tag{4.5}$$

Furthermore, the Euler formula gives:

$$2(b^0 + b^2 + b^3) + b_+^4 + b_-^4 = \chi \tag{4.6}$$

Using (4.5) and (4.6) together with the relation $\chi = 12(7 + p - q)$ we can easily determine:

$$b_-^4 = 3(h^{1,1} - 1) = 3p \tag{4.7}$$

That is, we end up with $n = 8h^{1,1} = 8(p + 1)$ right moving scalars which by $\mathcal{N} = (0, 6)$ supersymmetry have $8(p + 1)$ right moving Majorana fermions as superpartners. Again this is consistent with the fact that in the rigid supersymmetry limit the dimension of the target space of the right-moving moduli must be a multiple of 8.

Of course, to complete the spectrum we need to compute the left moving fields as well, but since they will play no part in the rest of the discussion we will not explicitly count the left-movers. Suffice it to say that they ensure that the resulting two dimensional theory is free from gravitational anomalies.

The $\mathcal{N} = (0, 6)$ supergravity coupled to $8n$ matter multiplets has not been constructed in literature. The important point about this theory is that the target space for the scalars is completely fixed, even though the theory has only six supercharges, it behaves more like the case of $\mathcal{N} = 4$ supergravity with 16 supercharges in four dimensions. There is a simple argument to see why the target space for the right-moving moduli is fixed by n , in the case of $\mathcal{N} = (0, 6)$ supergravity. It starts out with the observation that in the case of rigid supersymmetry, any sigma model with $\mathcal{N} = (0, 6)$ supersymmetry is based on a flat target space (upto orbifolding by a discrete group). The reason for this is simple: with $\mathcal{N} = (0, 6)$ supersymmetry and beyond, the only super multiplet possible with this much supersymmetry has scalars transforming non trivially under the R-symmetry that rotates the supercharges. In the $\mathcal{N} = (0, 8)$ case for example the scalars form a $\mathbf{8}_v$ of the spin(8) R-symmetry, whereas in the $\mathcal{N} = (0, 6)$ case the scalars form a $\mathbf{4}$ of $SU(4) \sim SO(6)$. Every such sigma model if it were non trivial would give rise to a conformally invariant theory in the IR, with $\mathcal{N} = 6$ SCA and above. However there is no superconformal extension of the $\mathcal{N} = 6$ supersymmetry algebra. This means the IR theory must be scale invariant without being conformally invariant and therefore every such sigma model should actually correspond to a free theory.⁴

Indeed in the $\mathcal{N} = (0, 6)$ case the scalars transform in the $\mathbf{4}$ of $SU(4)$ so there are actually $8n$ scalar fields rather than a multiple of 6 which would have required the scalars to transform in the fundamental of $SO(6)$. As we have argued above, in the case of rigid $\mathcal{N} = (0, 6)$ supersymmetry the target space parameterized by the right-moving scalars is actually flat and is simply \mathbf{R}^{8n} locally (in our case $n = p + 1$). This means that any non-trivial moduli space arises for these scalars precisely by coupling to $\mathcal{N} = (0, 6)$ supergravity. Upon coupling to $\mathcal{N} = (0, 6)$ supergravity there is a mass parameter κ that essentially plays the role of the gravitational Newton's constant in 4d (the 2d gravitational

⁴In the non compact case one can have scale invariance without conformal invariance essentially by turning on the dilaton, but this is not possible in the compact case.

coupling is dimensionless). When $\kappa \rightarrow 0$ the target space becomes flat \mathbf{R}^{8n} and for non-zero κ the target space for the right-moving moduli must have a curvature proportional to κ . All of this is analogous to what happens for $\mathcal{N} = 2$ supergravity coupled to matter in four dimensions. In this case in the rigid supersymmetry limit the target space for the scalars must be hyper-Kähler, whereas local supersymmetry requires the target space to be quaternionic Kähler with negative curvature proportional to the 4d Newton's constant. The only difference is that for $\mathcal{N} = (0, 6)$ supergravity in 2d, the rigid supersymmetry limit is trivial and this we expect will put severe constraints on the moduli space arising out of local supersymmetry. In particular, this moduli space can be exactly determined. The actual construction of $\mathcal{N} = (0, 6)$ supergravity coupled to matter will be explored in a forthcoming paper.

This allows us to resolve the puzzle raised at the end of the previous section. Even though supergravity does not drastically constrain the moduli space of type IIA compactifications to three dimensions (with six supercharges), it turns out that supergravity does constrain the moduli space of type IIB compactification. Since type IIA and IIB are related to each other upon compactifying one dimension further (and T-dualizing) this provides us with an understanding of why the type II moduli space for hyper-Kähler four-fold compactifications can be determined locally.

We claim that the type IIB moduli space \mathcal{M} is given by:

$$\mathcal{M} = \mathbf{T}^* \frac{O(4, p+1)}{O(4) \times O(p+1)} \tag{4.8}$$

That is locally \mathcal{M} is a bundle over the Grassmannian with fibers \mathbf{R}^{4p+4} . Globally of course, the fibers are compact, and are actually tori (which is what makes the moduli space compact, assuming the T-duality group acts on the base). The form of the moduli space we expect from the large radius limit is:

$$\mathcal{M} = \mathbf{R}^{5p+6} \times \frac{SL_2(\mathbf{R})}{U(1)} \times \frac{O(3, p)}{O(3) \times O(p)} \tag{4.9}$$

We have been schematic in writing (4.9) and it should be thought of as a warped product of the individual factors. Since the radial mode goes into the $\mathcal{N} = (0, 6)$ supergravity multiplet, we expect the large radius limit to be exact.

The moduli space at weak coupling (the CFT moduli space) is of the form:

$$\mathcal{M} = \mathbf{R}^{4p+4} \times \frac{O(4, p+1)}{O(4) \times O(p+1)} \tag{4.10}$$

which agrees with the topology of (4.8). One way to show (4.8) is to simply dimensionally reduce type IIB on hyper-Kähler four-folds in a manner similar to and observe that the moduli space is of the form of a cotangent bundle over the space parameterized by the metric and B-field deformations. The cotangent bundle structure follows exactly as in the analysis of Gates, Gukov and Witten . This structure of course arises rather straightforwardly upon dimensional reduction, but our claim is that this form of the moduli space is fixed by the $\mathcal{N} = (0, 6)$ supergravity of the 2d theory.

4.1 T-duality

The space (4.8) has a base which can be thought of as the space of space-like four planes in $\mathbf{R}^{4,p+1}$. The group $O(4, p + 1)$ naturally acts on $\mathbf{R}^{4,p+1}$ into which we can embed an integral lattice $\Lambda^{4,p+1}$. The subgroup $O(3, p)$ of $O(4, p + 1)$ is the rotation of the integral lattice Λ^{3,b_2-3} of H^2 . In type IIB there is also the $sl_2(\mathbf{Z})$ duality inherent in ten dimensions. This leaves the lattice Λ^{3,b_2-3} untouched but mixes the base and fibers. It was pointed out by Verbitsky [9] that there is a group action on the integral cohomology lattice of any hyper-Kähler $2n$ -fold of the form $SO(4, b_2 - 2)$ which in particular holds for four-folds. The $O(4, p + 1)$ factor can thus be identified with the symmetry of the integral cohomology lattice of the four-fold. This motivates the $O(4, p + 1; \mathbf{Z})$ duality group of type IIA. In type IIB this has to be extended by the action of $sl_2(\mathbf{Z})$.

In the type II theories D-brane charges are vectors in the lattice $H^*(X; \mathbf{Z})$ and the action of $SO(4, b_2 - 2)$ that acts as an automorphism of this lattice rotates the D-brane charges the way T-duality is supposed to work. This leads us to suspect that $SO(4, p + 1; \mathbf{Z})$ is nothing but the T-duality group of the type IIA theory.

Indeed $SO(4, p + 1; \mathbf{Z})$ is the T-duality group of the worldsheet SCFT corresponding to hyper-Kähler four folds. However as we will see soon some of these compactifications are destabilized by the 1-loop correction. In those cases the T-duality group may be strictly smaller. The fact that the classical T-duality group is $SO(4, p + 1; \mathbf{Z})$ ties in neatly with the observation that $SO(4, p + 1)$ acts on $H^*(X)$ via the fact that D-brane charges are Mukai vectors in the lattice $H^*(X; \mathbf{Z})$.

4.2 Dimension of the \mathcal{M} -theory moduli space

The dimension of the moduli space of type IIA (or \mathcal{M} -theory) compactification on hyper-Kähler four-folds is set by two integers p and q . However it is easy to show that there is an upper bound on p for any hyper-Kähler four-fold.

In fact, Beauville has shown that $b_2 \leq 23$ which implies $p \leq 20$. This restriction follows from the observation that $Sym^2(H^2) \hookrightarrow H^4$.

From this we note:

$$b_2(1 + b_2) \leq 2b_4 \tag{4.11}$$

Furthermore, by an Index theorem of Salamon (we have written it for hyper-Kähler four-folds, though the Index theorem holds for all hyper-Kähler manifolds):

$$b_4 = -b_3 + 10b_2 + 46 \tag{4.12}$$

Using (4.11) and (4.12) we get:

$$(b_2 - 23)(b_2 + 4) \leq 0 \tag{4.13}$$

which implies $b_2 \leq 23$. In the case where $b_2 = 23$ the inclusion map $i : Sym^{2i}(H^2) \hookrightarrow H^{4i}$ is exact and gives the only non vanishing Hodge numbers leading to the Hodge diamond of the Hilbert scheme.

Rather non-trivially even the integer q is bounded from above for hyper-Kähler four-folds by a number that depends on p [10]. Indeed the analysis of [10] concludes that

not every topological type of hyper-Kähler four-fold is possible. Either $b_2 = 23$ in which case the hyper-Kähler four-fold has the same Hodge diamond as $\text{Hilb}^2(K3)$, or $b_2 \leq 8$. Furthermore, for all $b_2 \leq 8$, b_3 is bounded above by a number that depends on b_2 .

It is clear that (4.11) and (4.12) also put a bound on q but the bounds derived in [10] are much stronger.

Including the moduli coming from the position of membranes, the entire moduli space is of bounded dimension. A similar situation arises for $\mathcal{N} = 4$ supersymmetric string compactifications in four dimensions, but it is nice to see the moduli space of theories with six supercharges is bounded by a calculable finite number. It would be very interesting to understand a more physical reason for the precise bound on p and q . Of course, this would follow from the $\mathcal{N} = 4$ worldsheet SCFT by analyzing the Elliptic genus, together with the CFT version of (4.11), but perhaps there is a more compelling reason, just as the dimension of the $\mathcal{N} = 4$ theories in 4d were restricted by the rank of the gauge group that could come out of the Heterotic string.

5. Effect of fluxes

In the previous section we considered hyper-Kähler compactifications where we set all RR fluxes to zero, and indeed only the metric degree of freedom was excited. It turns out that in general this does not lead to consistent string propagation. In fact, there is a 1-loop correction to the B-field equation of motion of type IIA, which in general requires the RR 4-form flux to be turned on to solve. This is simplest to understand in the \mathcal{M} -theory context where Witten [11] observed that the G -flux of \mathcal{M} -theory does not obey Dirac quantization condition. If we consider the periods of the G -flux on a spin four-fold, then Witten showed:

$$\left[\frac{G}{2\pi}\right] - \frac{1}{2}\lambda \in \mathbf{Z} \tag{5.1}$$

where $\lambda = \frac{1}{2}p_1$. In other words G obeys usual quantization condition precisely when λ is even. If however λ is odd, then it is clear from (5.1) that G can not be taken to vanish. We will see that for the two known hyper-Kähler four-folds, this condition is not satisfied, so they do not solve the string equations of motion until one turns on G -flux. The condition (5.1) is closely related to the tadpole condition in \mathcal{M} theory [12], where the Bianchi identity of G is corrected as:

$$d * G = \frac{1}{4\pi^2} G \wedge G - \frac{(p_1^2 - 4p_2)}{192} \tag{5.2}$$

In fact, (5.2) upon integrating over the internal manifold has to be integral (so that it can be canceled by adding membranes). However, it can be checked [11] that the class $[\frac{G}{2\pi}]$ is integral precisely when λ is even according to (5.2), confirming (5.1). For a complex four-fold with $c_1 = 0$:

$$(p_1^2 - 4p_2) = 8\chi, \tag{5.3}$$

so that (5.1) implies $G = 0$ is consistent only if χ is divisible by 24. As we already noted, for a hyper-Kähler four-fold:

$$\frac{\chi}{24} = \frac{1}{2}(7 + p - q) \tag{5.4}$$

Equation (5.2) is a tadpole for the gauge field C which couples to the membrane charge in \mathcal{M} -theory. The charge conservation relation following from (5.2) is:

$$n + \frac{1}{2(2\pi)^2} \int_X G \wedge G = \frac{\chi}{24} \tag{5.5}$$

As a result of (5.5) we need to include membranes or turn on G -flux to cancel the tadpole for C . The effect of turning on fluxes will be discussed later in this section, and we will now simply consider the effect of adding n membranes spanning the noncompact directions. There will now be additional moduli arising from the fact that the n membranes can be placed at arbitrary points. These moduli are trivial to count, and including them the \mathcal{M} -theory moduli space becomes a warped product of the form:

$$\mathcal{M}_{11d} = \frac{O(4, p+1)}{O(4) \times O(p+1)} \otimes Sym^n(X) \tag{5.6}$$

Therefore the effect of adding membranes is to increase the moduli. However the effect of turning on fluxes is quite the opposite.

In the large radius limit the conditions for preserving supersymmetry upon turning on G -flux was analyzed by Becker and Becker [13].

The result is that the G -flux must be primitive and of Hodge type (2, 2). The primitivity condition can be written as:

$$G \wedge J = 0 \tag{5.7}$$

where J is the Kähler form. This of course preserves only four supercharges, as the freedom to rotate the \mathbf{P}^1 of complex structures is broken by G . In order to preserve the full $\mathcal{N} = 3$ supersymmetry in the \mathcal{M} -theory compactification (5.7) must hold for all the complex structures on the four-fold X .

The condition for the form to be of type (2, 2) follows as usual from (upon imposing self-duality of G):

$$G \wedge \Omega = 0 \tag{5.8}$$

where Ω is the holomorphic 4-form. As the holomorphic four form Ω is simply the square of the holomorphic two form ω , the condition (5.8) is equivalent to:

$$G \wedge \omega = 0 \tag{5.9}$$

In general the space of primitive type (2,2) forms is not easy to describe. Given a generic HK 4-fold there may not be a non-trivial space of such forms. However for any HK 4-fold we can show the existence of at least one such form $G_0 = \omega \wedge \bar{\omega} - \frac{1}{2}J^2$. Due to the explicit appearance of ω in G_0 such a G -flux breaks $\mathcal{N} = 3$ supersymmetry down to $\mathcal{N} = 2$.

The primitivity of G_0 follows from the fact that $g_{ab}\omega_{bc} = \bar{\omega}_{ac}$. This can be seen as follows.

Any hyper-Kähler manifold M has a triplet of complex structures J^A .

Every such complex structure is a anti-symmetric rank-2 tensor map $J : TM \rightarrow TM$ which squares to -1 , that is $J_j^i J_k^j = -\delta_k^i$. Furthermore, the almost complex structure

defined by J is integrable, so that extend the definition of J from the tangent space to a point on M to the entire manifold M . On a hyper-Kähler manifold the triplet of complex structures J are required to satisfy the Lie algebra of $sp(1)$. That is:

$$J^A J^B = \epsilon^{ABC} J_C \tag{5.10}$$

Once a complex structure is picked, we can write $J_1 = J$ and $J_2 + iJ_3 = \omega$ and $J_2 - iJ_3 = \bar{\omega}$ and (5.10) becomes:

$$J_j^i \omega^{jk} = \bar{\omega}^{ik} \tag{5.11}$$

where (i, j, k) indices refer to the real coordinates on M . Passing to complex coordinates (a, b, c) we note that ω is of type $(2,0)$ with respect to J and J is of type $(1,1)$ and in appropriate coordinates it can be expressed as $J_b^a = ig_b^a$ where $g_{a\bar{b}}$ is the Kähler metric. This means:

$$J^{\bar{a}b} \omega_{bc} = \bar{\omega}_c^{\bar{a}} \tag{5.12}$$

Now consider $i_J G_0$. Using:

$$J_{\bar{a}b} \omega^{bc} \bar{\omega}^{\bar{a}c} = g^{c\bar{d}} \bar{\omega}_{\bar{a}\bar{d}} \bar{\omega}^{\bar{a}c} \tag{5.13}$$

and applying (5.10) we infer:

$$i_J \omega \wedge \bar{\omega} = J \tag{5.14}$$

That is:

$$i_J(\omega \wedge \bar{\omega} - \frac{1}{2} J^2) = 0 \tag{5.15}$$

Equation (5.15) implies $i_J G_0 = 0$. For a middle dimensional form like G_0 , $i_J G_0 = 0$ implies $J \wedge G_0 = 0$. That is G_0 is primitive as we claimed. Note that $\omega \wedge \bar{\omega}$ is linearly independent from J^2 for hyper-Kähler manifolds with quaternionic dimension greater than 1, so G_0 is not vacuous. For a $K3$ surface of course $\omega \wedge \bar{\omega}$ is a linear multiple of $J \wedge J$ as the space of 4-forms on a $K3$ surface is one dimensional.

It appears that the choice of J in G_0 fixes the Kähler structure of the manifold. As we will see, the choice of ω fixes the complex structure up to an overall scale $\lambda \in \mathbf{C}^*$ and the G -flux can be taken to be of the form μG_0 so that $\omega \rightarrow \lambda \omega$, $\mu \rightarrow |\lambda|^{-2} \mu$ leaves G_0 fixed if $J \rightarrow |\lambda| J$ so that there is still a residual ambiguity corresponding to the radial modulus which is not fixed by the choice of this flux.

If however G satisfies both (5.7) and (5.9) and is not of the form G_0 then it is automatically primitive with respect to the entire \mathbf{P}^1 of complex structures.

Turning on G flux of the form G_0 constrains the complex structure. In fact the complex structure is entirely fixed by the choice of G -flux in this case.

Given ω we write $\omega = \alpha + i\beta$ where $(\alpha, \beta) \in H^2(X; \mathbf{R})$. For a $K3$ surface the middle dimensional cohomology was the lattice \mathbf{Z}^{22} which was even, unimodular and had a quadratic form with signature $(3,19)$ associated to it. For a higher dimensional hyper-Kähler manifold most of this structure does not generalize. However, for any hyper-Kähler manifold X we can look at the cohomology $H^2(X; \mathbf{Z})$. There is a quadratic form q_X called

the Beauville-Bogomolov form⁵ associated to $H^2(X; \mathbf{Z})$ making it into a lattice Λ^{3,b_2-3} which is integral but not necessarily even or unimodular.

In the case of $\text{Hilb}^2(K3)$ the lattice $\Lambda^{3,20} = E_8 \oplus E_8 \oplus H^3 \oplus (-2\mathbf{Z})e$ and with a suitable normalization the Beauville-Bogomolov quadratic form is even and integral.

The Hodge-Riemann identities are:

$$q_X(\omega, \omega) = 0 \quad q_X(\omega, \bar{\omega}) > 0 \tag{5.16}$$

In terms of α and β we have:

$$q_X(\alpha, \alpha) = q_X(\beta, \beta), \quad q_X(\alpha, \beta) = 0 \tag{5.17}$$

That is, the choice of a holomorphic 2-form ω is equivalent to the choice of a space-like 2-plane spanned by the periods of α and β in the period domain Λ^{3,b_2-3} . Together with J this determines a space-like 3-plane \mathcal{O} in Λ^{3,b_2-3} .

The choice of flux G_0 is automatically a choice of ω and J so it corresponds to a choice of the space-like 3-plane \mathcal{O} inside Λ^{4,b_2-2} . By supersymmetry the periods of the B -field is also fixed by this choice of G -flux. However splitting off $\Lambda^{4,b_2-2} = \Lambda^{3,b_2-3} \oplus \Lambda^{1,1}$ we see that fixing the space-like 3-plane \mathcal{O} still leaves unfixed the radial mode (in fact one complex dimension if we include the $\mathcal{N} = 2$ superpartner) remains unfixed. It is still non-trivial to find a G_0 that satisfies the flux quantization condition. We will examine this in detail in the next section in the context of an example.

5.1 Extra constraints on moduli

In general, there are only three equations arising from the supergravity constraints (5.7) and (5.8). When the G -flux is not of type G_0 it appears therefore that only a triplet of moduli can be removed at each instance, whereas with $\mathcal{N} = 3$ supersymmetry we expect quaternionic dimensions to disappear. This means there is more moduli being removed than governed by (5.7) and (5.8). In the \mathcal{M} -theory setting this happens because certain modes of the 3-form C are constrained due to the Chern-Simons coupling $C \wedge G \wedge G$.

Indeed upon turning on G -flux this Chern-Simons coupling leads in a standard fashion to the 3d Chern-Simons action for the zero mode of C so that the G -flux appears to give topological mass to the vector field C_μ sitting in one of the p vector multiplets. Together with the mass terms for the triplet of scalars, this is enough to lift precisely one quaternionic dimension.

The 11d supergravity action is of the form:

$$S = \int d^{11}x \sqrt{-g} (R - \frac{1}{2} F \wedge *F) - \frac{1}{24\pi^2} C \wedge F \wedge F \tag{5.18}$$

Expanding the 3-form in harmonics:

$$C = \sum_a \omega_a A_\mu^a + \sum_i \omega_i^{2,1} A^i \quad \omega_a \in H^2(X; \mathbf{R}) \tag{5.19}$$

⁵An excellent introduction to compact hyper-Kähler manifolds is the lectures of Huybrechts [14] and the paper of Beauville [15] where the quadratic form is introduced. Integrality follows by a result of Fujiki [16].

we end up with a 3d Chern-Simons action:

$$S = \frac{1}{4\pi} \int d^3x \lambda_{ab} A^a \wedge F^b, \quad \lambda_{ab} = \int_X \omega_a \wedge \omega_b \wedge \frac{G}{\pi} \quad (5.20)$$

In the case where λ is not even, $\frac{G}{\pi}$ is an integral class, so (5.20) as normalized is $\frac{1}{2}$ of the canonical Chern-Simons action in 3d. Furthermore, λ_{ab} as defined in (5.20) is integral. The Chern-Simons action on a 3-manifold W is defined by computing the Maxwell action:

$$S = \frac{1}{2\pi} \int_Z F \wedge F \quad (5.21)$$

for an arbitrary closed 4-manifold Z with boundary W by choosing an extension of the gauge field on Z . The action (5.21) is independent of the choice of Z modulo 2π . Suppose the 4-manifold Z is spin, then if L is a complex line bundle over Z with $c_1(L) = \frac{F}{2\pi}$ then $c_1^2(L)$ is divisible by 2 by Wu's formula (as the second Steiffel-Whitney class w_2 vanishes). That is, given a 3-manifold with a chosen spin structure, the Chern-Simons action:

$$S = \frac{1}{4\pi} \int A \wedge F \quad (5.22)$$

is the basic action (the so called level- $\frac{1}{2}$ Chern-Simons action). This agrees with the normalization in (5.20).

Going back to (5.20), we notice that the effect of turning on background G -flux is to give topological mass to the gauge fields. By $\mathcal{N} = 3$ supersymmetry (5.20) is related to mass terms for the $\mathcal{N} = 3$ superpartners. The $\mathcal{N} = 3$ vector multiplet in 3d consists of a vector and three scalars. In $\mathcal{N} = 2$ notation we write the $\mathcal{N} = 2$ vector multiplet as Σ and the chiral multiplet as Φ . Σ contains a real scalar and a vector which together with the chiral multiplet form the content of a $\mathcal{N} = 3$ vector multiplet (the theory is parity invariant under $\lambda_{ab} \rightarrow -\lambda_{ab}$ so the multiplet is the same as a $\mathcal{N} = 4$ vector multiplet). In terms of this the superpotential can be schematically written as:

$$S = \int d^3x d^4\theta \lambda_{ab} \Sigma^a V^b - \int d^3x d^2\theta i \lambda_{ab} \Phi^a \Phi^b, \quad \Sigma = iD\bar{D}V \quad (5.23)$$

(5.23) is only schematic since we have ignored the coupling to gravity and as written (5.23) is simply the $\mathcal{N} = 3$ supersymmetric Chern-Simons action.

For non-zero G , parity invariance in 3d is broken by the Chern-Simons coupling (5.20). This is simply because the G -flux is odd under 11d parity, and any expectation value breaks parity in 11d, and upon compactification in the resulting 3d theory also.

The analysis leading to (5.20) is really independent of the details of the internal manifold which are subsumed in λ_{ab} . Suppose we consider $K3 \times K3$. In this case the 3d theory has $\mathcal{N} = 4$ supersymmetry which prevents the appearance of a Chern-Simons term. However, as shown in [5] it is possible to turn on G -flux consistent with $\mathcal{N} = 4$ supersymmetry. Indeed as we saw above, the 11d Chern-Simons coupling gives rise to a topological mass for the gauge fields irrespective of the precise amount of supersymmetry, so we should expect this coupling to be present even for $K3 \times K3$. However, it is well known that there is

no $\mathcal{N} = 4$ supersymmetric Chern-Simons action. There is in fact only one way to complete (5.20) in a manner consistent with $\mathcal{N} = 4$ supersymmetry. To explain this let us consider the dimensional reduction on $K3 \times K3$.

Upon dimensionally reducing (5.18) the gauge fields A_μ^a arise via reduction of C on $K3 \times K3$. An equal number of such gauge fields arise via dimensional reduction on either $K3$. Instead of considering all those gauge fields together as $2(h^{1,1} + 2)$ vector multiplets of the $\mathcal{N} = 4$ supersymmetry, we can rather consider them as $h^{1,1} + 2$ vector multiplets and $h^{1,1} + 2$ twisted vector-multiplets. Doing so the 11d Chern-Simons coupling leads upon dimensional reduction to a BF type coupling between the vector and twisted vector-multiplets, lifting a pair of quaternionic dimensions at a time.

To be more precise, the $\mathcal{N} = 4$ supersymmetry algebra in three dimensions has a $SU(2)_R \times SU(2)_N$ R-symmetry, the eight super charges being doublets under the two R-symmetry factors. The $\mathcal{N} = 4$ vector multiplet has a vector, three real scalars transforming as $\mathbf{3}$ of $SU(2)_R$ as bosonic components. The $\mathcal{N} = 4$ supersymmetry algebra admits an automorphism that exchanges the two $SU(2)$ factors and takes a vector multiplet into a so-called twisted vector-multiplet which has three scalars that transform as $\mathbf{3}$ of $SU(2)_N$. In the $\mathcal{N} = 4$ supergravity that arises upon compactifying \mathcal{M} -theory on $K3 \times K3$, the two $SU(2)$ factors can be related to the holonomies of the $K3$ s. In fact, upon compactifying \mathcal{M} -theory on a product of four-manifolds $Y \times Y$, the holonomy group is $SO(4) \times SO(4)$ generically, leading to the absence of R-symmetries in the resulting 3d theory (which is not supersymmetric unless Y has reduced holonomy). Suppose Y is a $K3$ surface, then decomposing $SO(4)$ as $SO(4) = SU(2) \times SU(2)$ the holonomy of Y can be taken to be one of the two $SU(2)$ factors, and the other $SU(2)$ factor therefore becomes an R-symmetry. The same thing happens with the other factor of Y thus leading to a $SU(2)_R \times SU(2)_N$ R-symmetry as noted. The important point is that the two $SU(2)$ factors are associated with the two $K3$ surfaces.

With this identification, it is clear that the $2(h^{1,1} + 2)$ vector multiplets that arise by dimensional reduction of the 3-form have to be treated as $(h^{1,1} + 2)$ vector multiplets and $(h^{1,1} + 2)$ twisted vector-multiplets as claimed, because the scalars in these multiplets transform under different R-symmetries. There is a unique renormalizable coupling that involves vector and twisted vector-multiplets and is called the BF coupling [17]. It is precisely this coupling that arises via dimensional reduction of \mathcal{M} -theory to 3d.

Again schematically the $\mathcal{N} = 4$ superpotential can be written as:

$$S = \int d^3x d^4\theta \lambda_{aa'} \Sigma^a \tilde{V}^{a'} - i \lambda_{aa'} \Phi^a \tilde{\Phi}^{a'} \tag{5.24}$$

where Σ, Φ form a $\mathcal{N} = 4$ vector multiplet and $\tilde{\Sigma}$ and $\tilde{\Phi}$ form a $\mathcal{N} = 4$ twisted-vector multiplet.

6. Examples

Examples of compact hyper-Kähler four-folds are very hard to obtain. There are only two known examples in literature, and both of them are obtained from symmetric products of

complex 2-folds (a K3 surface in the case of the Hilbert scheme, and T^4 in the example of Beauville).

6.1 The Hilbert scheme of two points on K3

A K3 surface is a compact (simply connected) Kähler surface with trivial canonical bundle (its holonomy is $SU(2) = sp(1)$ and is in fact a hyper-Kähler manifold of complex dimension 2). As all K3 surfaces are diffeomorphic, one can compute the unique Hodge numbers of a K3 surface by picking a suitable representative. The Fermat form of the quartic in \mathbf{CP}^3 is a simple example given by:

$$S : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \quad z_i \in \mathbf{C} \quad z_i \sim \lambda z_i \quad \lambda \in \mathbf{C}^* \tag{6.1}$$

By the adjunction formula one can compute the Chern classes of the surface S using its embedding in \mathbf{CP}^3 :

$$c(T_S)c(N_S) = c(T\mathbf{CP}^3|_S) \tag{6.2}$$

Using the fact that

$$c(T\mathbf{CP}^3|_S) = (1+x)^4 \quad c(N_S) = (1+4x) \quad x \in H^2(\mathbf{CP}^3, \mathbf{Z}) \tag{6.3}$$

We compute $c_1(S) = 0$ and $c_2(S) = 6x^2$. The Lefschetz hyperplane theorem tells us S is simply connected, so S is actually a K3 surface as advertised. Its Euler characteristic is:

$$\chi = \int_S c_2 = 24 \int_{\mathbf{CP}^3} x^3 = 24 \tag{6.4}$$

The Euler characteristic also has an expression in terms of Hodge numbers as:

$$\chi = h^{1,1} + 2h^{2,0} + 2 \tag{6.5}$$

allowing us to read off $h^{1,1} = 20$, since we know $h^{2,0} = 1$.

Given a K3 surface X , there is a construction by Beauville,⁶ that allows us to obtain a compact hyper-Kähler four-fold. Starting from a K3 surface X , the symmetrized product $S^2(X)$ admits a symplectic structure that is derived from the holomorphic 2-form present on X .

However, as it stands there are other symplectic structures also, and we need to be able to construct a unique (upto scaling) symplectic structure to obtain a hyper-Kähler 4-fold. This can be done by considering $X^2 = S^2(X)/G_2$ where G_2 is the symmetric group of order 2. This space is singular, but has a nice de singularization into what is called a Hilbert scheme $X^{[2]}$ which parameterizes finite subspaces of X of length 2. The Hilbert scheme can be thought of as a resolution of the singular space X^2 , and the holomorphic 2-form survives this resolution and yields a smooth, compact hyper-Kähler four fold $X^{[2]}$.

One can determine $b_2(X^{[2]}) = b_2(X) + 1$ and $b_3(X^{[2]}) = 0$, so that the Hodge numbers of $X^{[2]}$ can be determined as follows:

$$h^{1,1} = 21 \quad h^{2,1} = 0 \quad h^{2,2} = 232 \tag{6.6}$$

⁶A nice introduction to hyper-Kähler manifolds can be found in Beauville's lectures [18]

This allows us to conclude that $\frac{\chi}{24} = \frac{27}{2}$, so we need to turn on G -flux to satisfy the G -equation of motion in \mathcal{M} -theory. This implies a destabilization of the purely gravitational background. We need to turn on G -flux, and furthermore if we want to preserve any supersymmetry the G -flux has to be primitive.

As we have mentioned before, the form $\frac{G}{\pi} = \mu G_0$ is primitive. Inserting the form of this flux into (5.5) we obtain:

$$n + \frac{1}{8}\mu^2 \int_X (G_0/\pi)^2 = \frac{27}{2} \tag{6.7}$$

We also require:

$$\frac{G}{\pi} \in H^4(X; \mathbf{Z}) \tag{6.8}$$

Only if (6.7) and (6.8) simultaneously have solutions do we have a consistent string background at this order. To preserve supersymmetry n has to be non-negative.

Let us analyze this more carefully.

First of all, let us fix the normalization of G_0 to make it primitive. It is clear that $i_J(\omega \wedge \bar{\omega})$ is proportional to J^2 so we need to fix the constant of proportionality. For this we can use the fact that:

$$\int_X J^4 = 3q_X^2(J, J) \quad \int_X \omega \bar{\omega} J^2 = q_X(\omega, \bar{\omega}) q_X(J, J) \tag{6.9}$$

so that:

$$\frac{G}{\pi} = \mu(\omega \wedge \bar{\omega} - \frac{q_X(\omega, \bar{\omega})}{3q_X(J, J)} J \wedge J) \tag{6.10}$$

is primitive and type (2,2). Now it is easy to compute (6.7) to be:

$$n + \frac{5}{24}\mu^2 q_X^2(\omega, \bar{\omega}) = \frac{27}{2} \tag{6.11}$$

which has a solution with $n = 6$ and $q_X(\omega, \bar{\omega}) = 2$ and $\mu = \pm 3$.

Flux quantization required $\omega \wedge \bar{\omega}$ to be integral (we simply absorb μ into ω by scaling $\omega \rightarrow \lambda \omega$ with $\lambda = \sqrt{\mu}$).

As mentioned before, the HK moduli space is lifted save for the complex dimension in which the radial mode resides. There are however moduli arising from the position of the membranes in the transverse eight dimensional space.

The above analysis raises the following puzzle: in the orbifold limit of $K3^{[2]}$ studied by Dasgupta, Rajesh and Sethi [19], it was found that there was no way of turning on G -flux in a supersymmetric manner for the orbifold $K3^2$ whereas we have just argued that $K3^{[2]}$ does have a supersymmetric solution upon turning on suitable G -flux.

Let us try to locate the precise nature of this discrepancy. In [19] the orbifold limit of the symmetric product of $K3$ was considered. In the limit where $K3$ could be described by T^4/\mathbf{Z}_2 as:

$$g_1 : (z^1, z^2) \rightarrow -(z^1, z^2) \tag{6.12}$$

They obtained the condition for a primitive type (2, 2) G -flux to solve the anomaly cancellation condition as:

$$2(|A|^2 + B^2 + C^2) + n = \frac{27}{2} \tag{6.13}$$

where the primitive type (2,2) flux was written down as:

$$\frac{G}{2\pi} = Ad\bar{z}^1 dz^2 d\bar{z}^3 dz^4 + Bd\bar{z}^1 dz^2 dz^3 d\bar{z}^4 + Cd\bar{z}^1 d\bar{z}^2 dz^3 dz^4 + h.c \quad (6.14)$$

The condition that the period of $\frac{G}{2\pi}$ be half integral quantized gave the restriction:

$$(ReA \pm B \pm C) \in \mathbf{Z} \quad ImA \in \mathbf{Z} \quad (6.15)$$

Due to the symmetrization (B, C) are real. Now it was noted in [19] that there is no solution to both (6.13) and (6.15). This means the orbifold limit of $K3^{[2]}$ is not a supersymmetric solution. This seems to directly contradict our claim that $K3^{[2]}$ preserves supersymmetry as a type IIA or \mathcal{M} -theory compactification. Indeed there is something puzzling about the fact that (6.13) and (6.15) do not agree.

To be precise, any solution to the flux quantization condition allows us to solve (6.15) for an integral number of branes. In the orbifold limit $S^2(\mathbf{T}^4/\mathbf{Z}_2)$ the flux quantization condition simply requires the periods of $\frac{G}{2\pi}$ to be integral (or half-integral). Away from the quotient singularities the space is locally flat, and λ can be taken to vanish.

In the absence of torsion, this would imply the quantization condition (6.15). Now in the case of compact eight-manifolds, it was shown by Witten that any G flux solving the flux quantization condition also solves the anomaly cancelation condition with integral number of branes. This statement does not carry over to the orbifold as we just saw above. This is somewhat surprising.

As we saw above, we can turn on G -flux with $\mathcal{N} = 2$ supersymmetry preserving vacua. In this case we lift all the complex structure moduli, and we're left with one Kähler moduli and the associated period of the B-field. That is there is a \mathbf{R} worth of B-field periods. Now $S^2(K3)$ with G -flux turned on has atleast one B-field period that is unfixed, this corresponds to the volume modulus of the $K3$ itself. We will see this to imply that the orbifold point $S^2(K3)$ does not exist in the $\mathcal{N} = 2$ moduli space.

To be more precise, we have to formulate what it means to reach the symmetric product point $S^2(K3)$. The moduli space of complex structures on $Hilb^2(K3)$ has dimension 20, which is one greater than the moduli space of complex structures of $K3$. This means at a generic point in moduli space the internal manifold is not of the form $K3^{[2]}$. Geometrically as we mentioned before, $K3^{[2]}$ is obtained from $S^2(K3)$ by blowing up the exceptional divisor $e \in H^{1,1}(K3^{[2]}; \mathbf{R})$. In fact $q_X(e, e) = -2$ so e is a time-like vector in $\Lambda^{3,20}$. A choice of complex structure is the same as a choice of a positive 3-plane in $\Lambda^{3,20}$ and this induces a polarization of e as $e = e_+^{3,0} + e_-^{0,20}$ where the \pm serve to indicate the projection into space-like and time-like parts. To reach the point in moduli space where we have $S^2(K3)$ we need to ensure that e is orthogonal to the 3-plane spanned by (J, ω) . In other words, one has to rotate the 3-plane spanned by (J, ω) such that it is orthogonal to e . However with $\mathcal{N} = 2$ supersymmetry preserving G -flux turned on, the complex structure is entirely frozen. Indeed the 3-plane spanned by (J, ω) is fixed by the choice of G_0 so it is no longer possible to reach the symmetric product point.

Let us determine solutions with $\mathcal{N} = 3$ supersymmetry. Let us first choose $G = \nu\alpha^2$ where α is an element of $H^{1,1}(X; \mathbf{Z})$.

This can be shown as follows: Pick primitive (2,2) form G as:

$$\frac{G}{\pi} = \nu\alpha \wedge \alpha = \nu\alpha^2 \quad \nu \in 2\mathbf{Z} + 1 \tag{6.16}$$

where $\alpha \in H^2(X; \mathbf{Z}) \cap H^{1,1}(X; \mathbf{R})$ is primitive. This means that α is orthogonal to the 3-plane spanned by (ω, J) in the lattice Λ^{3, b_2-3} and is furthermore time-like.

The anomaly cancelation condition now becomes:

$$n + \frac{\nu^2}{8} \int_X \alpha^4 = \frac{27}{2} \tag{6.17}$$

We can re-write (6.17) as:

$$n + 3\frac{\nu^2}{8} q_X^2(\alpha, \alpha) = \frac{27}{2} \tag{6.18}$$

As q_X is even, we can write $q_X(\alpha, \alpha) = 2k$ so that we are searching for solutions to:

$$n + \frac{3}{2}\nu^2 k^2 = \frac{27}{2} \tag{6.19}$$

Clearly there is a non vanishing space of solutions to (6.19). One such solution without any membranes has $k = 1$ and $\nu = \pm 3$.

So far we have not checked whether α^2 is primitive. That is, we require $i_J \alpha^2 = 0$. This actually cannot happen for our choice of G for any α .

In general we can write:

$$i_J \alpha^2 = \gamma \tag{6.20}$$

Further, we can consider:

$$\frac{G}{\pi} = \alpha^2 - J \wedge \gamma - \frac{1}{2} a J \wedge J \tag{6.21}$$

where we denote by a the contraction $\alpha^{i\bar{j}} \alpha_{i\bar{j}}$.

It is easy to see that $i_J G = 0$ if G is defined as in (6.21). This allows G to be a primitive type (2,2) form. However there is an explicit dependence on J so the choice of this form does not leave us with the freedom to rotate the space-like 3-plane \mathcal{O} , which means this choice also breaks $\mathcal{N} = 3$ supersymmetry.

Let us therefore consider a G -flux of the form $\frac{G}{\pi} = \alpha \wedge \beta$ with (α, β) in $H^{1,1}(X; \mathbf{Z})$ and orthogonal to \mathcal{O} . Such (α, β) lie in the Picard lattice of X . Under what condition is G primitive? Clearly when $\alpha = \beta$ such a G -flux cannot be primitive as we saw above. However, supposing α and β are linearly independent, is it possible to arrange for the G -flux defined above to be primitive?

In fact, consider $\alpha \wedge \beta \wedge J = A$. Now suppose $x \in H^2(X; \mathbf{R})$ be an arbitrary form, then we can show that $\int_X A \wedge x = 0$ which automatically implies $A = 0$ that is G is primitive if and only if α and β are orthogonal with respect to q_X . That is, a G -flux of the form:

$$\frac{G}{\pi} = \alpha \wedge \beta, \quad \alpha, \beta \in H^{1,1}(X; \mathbf{R}) \cap H^2(X; \mathbf{Z}), \quad q_X(\alpha, \beta) = q_X(\alpha, J) = q_X(\beta, J) = 0 \tag{6.22}$$

is primitive and of type (2,2) and turning on such a flux preserves $\mathcal{N} = 3$ supersymmetry.

That is, pick α and β in $\Lambda^{3,20}$, mutually orthogonal, and orthogonal to \mathcal{O} . Given such α and β the G -flux defined in (6.22) is primitive and type (2,2) and preserves $\mathcal{N} = 3$ supersymmetry as the choice of such a G -flux does not affect the rotations in \mathcal{O} .

In order to solve the anomaly cancellation condition we require:

$$n + \frac{1}{8}q_X(a, \alpha)q_X(\beta, \beta) = \frac{27}{2} \tag{6.23}$$

There are many solutions to (6.23) once we note that $\Lambda^{3,20}$ is an even lattice so (6.23) always has solutions.

For $\text{Hilb}^2(K3)$ the space H^4 is comprised entirely of $\text{sym}^2(H^2)$ so there are no other possibilities for G -flux that give rise to $\mathcal{N} = 3$ supersymmetry.

Turning on a G -flux of the form (6.22) reduces the dimension of the moduli space. In fact, the moduli space is:

$$\mathcal{M}_V = \frac{O(4, 19)}{O(4) \times O(19)} \tag{6.24}$$

We can find more possibilities by considering more general G -flux as a linear combination of pairs of the form (6.22) and each time we reduce the dimension of the moduli space by two quaternionic dimensions at a time.

The fact that there exist compactifications with $\mathcal{N} = 3$ supersymmetry forces us to reconsider the puzzle posed earlier. In this case, it seems possible to reach the symmetric product point while preserving $\mathcal{N} = 3$ supersymmetry, which seems to contradict the results of [19]. A possible resolution to this may be the following: the symmetric product treated using usual orbifold techniques requires a certain θ angle to be turned on. In the limit discussed in [19], this was implicit. If suppose the θ angle is fixed at zero upon turning on G -flux, then we cannot reach the limit discussed in [19].

6.2 Singularities

It is an interesting problem to classify the type of singularities that occur in the moduli space of hyper-Kähler four-fold compactifications. In the case of $K3$ surfaces it is well known that the only singularities that can occur in the $K3$ moduli space are orbifold singularities which have an ADE classification. An analogous understanding of higher dimensional hyper-Kähler manifolds is lacking. Following its definition in the $K3$ case we can define the Picard lattice of a HK 4-fold X as:

$$\text{Pic}X = H^2(X; \mathbf{Z}) \cap H^{1,1}(X; \mathbf{Z}) \tag{6.25}$$

Suppose we have an element α of the lattice $\Lambda^{3,20}$ of $K3^{[2]}$ which is orthogonal to \mathcal{O} , the 3-plane spanned by (J, ω) . Then α lies in $\text{Pic}(K3^{[2]})$. A generic $K3^{[2]}$ has a vanishing Picard group, so the existence of such α is a restriction on the complex structure of $K3^{[2]}$. Every element α of the Picard lattice gives rise to a line bundle L over $K3^{[2]}$ with $c_1(L) = \alpha$. Indeed the Picard group is nothing but the group of such line bundles with the product given by Whitney product formula. Any such line bundle is Poincare dual to a divisor L . The zero section of such a line bundle will describe a divisor which we also call L . The

existence of a zero section of the line bundle L can sometimes be figured out using the Riemann-Roch formula [20]:

$$\chi(L) = \frac{1}{2}(\frac{1}{2}q_X(\alpha, \alpha) + 3)(\frac{1}{2}q_X(\alpha, \alpha) + 2) \tag{6.26}$$

For line bundles over surfaces $\chi = h^0(L) - h^1(L) + h^2(L)$ and if $\chi > 0$ this implies L (or L^{-1}) has a section. This is no longer the case for four-folds. It would be interesting to understand when a section exists. If it does, the corresponding divisor will have zero volume. The volume of such a divisor L is $\int_L J^3$ which is nothing but $\int_X J^3 \wedge \alpha$. However, we can easily show that $\int_X J^3 \wedge \alpha = q_X(\alpha, J)q_X(J)$ upto an irrelevant numerical constant. Now $q_X(\alpha, J) = 0$ so this implies volume of L vanishes. Such choices of α will then lead to singular HK 4-folds.

It is well known that $Pic(K3^{[2]}) = Pic(K3) \oplus \mathbf{Z}e$. This means that the Picard lattice of $K3^{[2]}$ can non-trivial if the underlying $K3$ itself is of a special type.

The Picard number ρ of a $K3$ surface is the rank of the Picard lattice and $K3$ surface is called attractive if $\rho = 20$. Defining the lattice $\Upsilon = (H^{2,0}(K3) \oplus H^{0,2}(K3)) \cap H^2(K3; \mathbf{Z})$, for a generic $K3$ surface Υ is completely trivial but can have maximal rank 2. The transcendental lattice is defined as the orthocomplement of the Picard lattice in $H^2(K3; \mathbf{Z})$ and precisely when the rank of Υ is two, Υ coincides with the transcendental lattice and the $K3$ surface will be attractive.

One can define an analogous lattice Υ for hyper-Kähler four-folds also. From our discussion of $\mathcal{N} = 2$ supersymmetry preserving fluxes on $K3^{[2]}$ we see that α and β can be expanded in an integral basis e_k for $H^2(K3^{[2]}; \mathbf{Z})$ as:

$$\alpha = \sum_k a_k e_k \quad \beta = \sum_l b_l e_l \quad \gamma = \sum_j c_j e_j \quad j, k, l = 1, \dots, 23 \tag{6.27}$$

Furthermore we can write $\frac{G}{\pi} = \sum_{kl} N_{kl} e_k \otimes e_l$ where we have the relation:

$$N_{kl} = 2a_k b_l + 2a_l b_k - c_k c_l \tag{6.28}$$

Here a_k and b_l are apriori real numbers and N_{kl} are integers (due to the flux quantization condition on G). Fixing l in (6.28), the equation (6.28) implies that a real linear combination of α , β and γ lies on a lattice point of $H^2(K3^{[2]}; \mathbf{Z})$. Varying l we get 23 such possibilities. As α , β and γ are linearly independent this implies the rank of Υ is precisely 2. That means the corresponding $K3^{[2]}$ is what we would call attractive by analogy with the $K3$ case.

Therefore, just as for $K3 \times K3$ turning on $\mathcal{N} = 2$ supersymmetry preserving G -flux leads to points in moduli space where the underlying $K3^{[2]}$ is attractive. In the case of $K3^{[2]}$ it is natural to associate the attractive $K3^{[2]}$ with the attractive $K3$ from which $K3^{[2]}$ is obtained.

In fact, turning on G -flux of the form G_0 restricted the complex structure and Kähler moduli of the hyper-Kähler four-fold such that the complex structure was entirely fixed, and the Kähler structure was essentially fixed upto scaling. There is a canonical way [21] to associate an attractive $K3$ surface S to a hyper-Kähler four-fold of the form $K3^{[2]} \simeq X$.

Pick a Mukai vector v in $H^*(S; \mathbf{Z})$ with $v^2 > 0$ in the lattice $\Lambda^{3,19}$ of the $K3$ surface. Then the Mukai map restricted to v^\perp gives an isomorphism between lattices of v^\perp and $H^2(X; \mathbf{Z})$ where X is the moduli space of semi-stable torsion free sheaves on $K3$ surfaces. For $v^2 = 2$, X coincides with the Hilbert scheme $K3^{[2]}$ upto a HK deformation. The positive 2-plane P in $H^2(X; \mathbf{R})$ spanned by the real and imaginary parts of the holomorphic 2-form ω on X is identified with the positive 2-plane of $K3$, so the complex structure of X is determined entirely by the complex structure of the corresponding attractive $K3$. Including the periods of the B -field on $K3$ determines the Kähler structure of X also, as shown in [21].

For this reason it is not surprising that turning on $\mathcal{N} = 2$ supersymmetric fluxes lead to a Υ that can be associated with an attractive $K3$ surface.

6.3 Beauville’s example [15]

If X is complex torus of dimension two, then the generalized Kummer variety $K^n(X)$ is an irreducible, holomorphic symplectic manifold of complex dimension $2n$. For $n = 2$ the corresponding hyper-Kähler 4-fold has $b_2 = 7$ and we expect a moduli space of the form (ignoring the moduli associated with the $h^{2,1}$ moduli):

$$\mathcal{M}_V = \frac{O(4, 5)}{O(4) \times O(5)} \tag{6.29}$$

in the absence of fluxes. As $h^{2,1} = 2$ the Euler number can be computed to be $\frac{\chi}{24} = \frac{9}{2}$. Thus we need to turn on flux even for these backgrounds. Let us look for solutions with $\mathcal{N} = 3$ supersymmetry.

We need to solve:

$$n + \frac{\nu^2}{8} \int_X (\alpha \wedge \beta)^2 = \frac{9}{2} \quad \nu = 2\mathbf{Z} + 1 \tag{6.30}$$

For $K^2(T^4)$ the normalization is such that:

$$\int_X \alpha^4 = 9q_X^2(\alpha, \alpha) \quad \int_X (\alpha \wedge \beta)^2 = 3(2q_X^2(\alpha, \beta) + q_X(\alpha)q_X(\beta)) \tag{6.31}$$

This implies:

$$\int_X (\alpha \wedge \beta)^2 = 12p \quad p \in 2\mathbf{Z} + 1 \tag{6.32}$$

and:

$$n + \frac{3}{2}\nu^2 p = \frac{9}{2} \tag{6.33}$$

This has a solution $\nu = \pm 1$ and $n = 0$, where $q_X(\alpha, \beta) = 0$ and $q_X(\alpha) = -6, q_X(\beta) = -2$.

As in the case of $\text{Hilb}^2(K3)$ we need to impose the condition that G -flux is primitive in order for solutions to preserve supersymmetry.

Picking a G -flux in $\text{sym}^2(H^2)$ to be of the form (6.22) we can solve the anomaly cancelation condition as in the previous section. Again we find $\mathcal{N} = 3$ supersymmetric solutions with a moduli space of the form:

$$\mathcal{M}_V = \frac{O(4, 3)}{O(4) \times O(3)} \tag{6.34}$$

However, unlike $\text{Hilb}^2(K3)$ it is no longer true for $K^2(T^4)$ that all of H^4 is generated from $\text{sym}^2(H^2)$. This allows us to consider a priori a G -flux of the form $\frac{G}{\pi} = x$ with $x \in (H^4(X; \mathbf{R}) \setminus \text{sym}^2(H^2(X; \mathbf{R}))) \cap H^{2,2}(X; \mathbf{Z})$.

With such a G -flux we need to solve:

$$n + \frac{1}{8} \int_X (x^2) = \frac{9}{2} \tag{6.35}$$

In order for there to be solutions to (6.35) we need $\int_X x^2$ to be divisible by 4. As long as $x \in \text{sym}^2 H^2$ this divisibility is guaranteed by the fact that the Beauville-Bogomolov form is even, for $K^2(T^4)$.

Suppose there is such a x which solves the anomaly cancellation condition. Then turning on a G -flux proportional to x does not lift any vector multiplet moduli. Indeed the simplest way to see this is of course that the moduli space \mathcal{M}_V was associated to $H^2(X)$ and the G -flux does not depend on H^2 . Another way to see it is to note that given a x in the ortho-complement of $\text{sym}^2(H^2)$ in H^4 , it is automatically primitive. However, if x belongs in the ortho-complement to $\text{sym}^2(H^2)$ (the inner product being given by the intersection form on H^4) then the intersection numbers λ_{ab} vanish. This means turning on such a G -flux does not lift any moduli and leads to a solution with $\mathcal{N} = 3$ supersymmetry and a moduli space identical to (6.29) except for possible membranes on \mathbf{R}^3 . Furthermore, it is not possible to lift any hyper-multiplet moduli by turning on such a flux.

The only question to address is whether such x satisfies the anomaly cancellation condition. For $K^2(T^4)$, c_2 belongs to $\text{sym}^2(H^2)$ so that $\lambda \cdot x = 0$. Now Wu's formula tells us $x^2 = \lambda \cdot x \pmod{2}$ so that x^2 is even.

We do not know how to evaluate if x^2 can be divisible by 4. It is possible that there is such an x for $K^2(T^4)$ in which case we would have another $\mathcal{N} = 3$ supersymmetric solution. The somewhat surprising thing here would be that the flux so turned on preserves $\mathcal{N} = 3$ supersymmetry but does not lift any moduli at all. We believe this unlikely, so we suspect such a flux x cannot be of type (2,2).

In order to determine solutions with $\mathcal{N} = 2$ supersymmetry we follow the discussion in the previous section and turn on G -flux of the form (6.10). The anomaly cancellation condition now becomes:

$$n + \frac{5}{8} q_X^2(\omega, \bar{\omega}) = \frac{9}{2} \tag{6.36}$$

and flux quantization requires $\omega \in H^2(X; \mathbf{Z}) \cap H^{2,0}(X; \mathbf{C})$ and $\sqrt{\frac{q_X(\omega, \bar{\omega})}{3q_X(J, J)}} J \in H^{1,1}(X; \mathbf{Z})$.

There is a solution to (6.36) which requires $n = 2$ and $q_X(\omega, \bar{\omega}) = 2$.

7. Discussion

In this note, we have discussed the moduli space of hyper-Kähler four-fold compactifications in type IIA/B and \mathcal{M} -theory. As we have seen, it is possible to put a strict upper bound on the dimension of the moduli space of hyper-Kähler four-fold compactifications. It should not be too surprising that the moduli spaces coming out of string theory are bounded: after all they are related to the moduli space of the worldsheet SCFT and we have every reason

to expect this moduli space to be finite dimensional. What is perhaps a little surprising is that one has a strict upper bound as a mathematical result. It would be very interesting to obtain this result from string theory arguments, or find a more intuitive reason for this bound.

One is free to think that hyper-Kähler four-fold compactifications only parameterize a small part of the space of all $\mathcal{N} = 3$ vacua arising from string theory so these bounds cannot be taken to imply anything about the bound on the moduli space of $\mathcal{N} = 3$ vacua in string theory. We try to construct other theories with six supercharges from string theory in appendix 2, and show that if anything the space of $\mathcal{N} = 3$ supersymmetric string vacua is very tightly constrained. There are no known large class of compactifications that achieve $\mathcal{N} = 3$ supersymmetry with or without fluxes. The possible exception to this may be the compactifications discussed in [22].

As emphasised in [23] the fact that the scalar moduli spaces arising in string theory are typically finite provides us a clue that consistent coupling of matter to gravity is a very nontrivial problem. From whatever we know about low energy supergravity there appears to be no reason for these moduli spaces to be bounded.

More work is needed to determine if there are some solutions of orbifold/ orientifold type which do yield $\mathcal{N} = 3$ supersymmetry upon turning on fluxes. If such solutions exist, they will be expected to belong to the moduli space of hyper-Kähler compactifications and may perhaps allow us to understand this moduli space better.

We have also analysed the two known examples of compact hyper-Kähler four-folds and found that in the case of $\text{Hlb}^2(K3)$ there is an interesting subtlety associated with the orbifold limit $S^2(\mathbf{T}^4/\mathbf{Z}_2)$.

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A. Worldsheet aspects of hyper-Kähler four-folds

The worldsheet description of $\mathcal{N} = (3, 3)$ and $\mathcal{N} = (0, 6)$ supersymmetric string compactifications to two dimensions starts with an internal $\mathcal{N} = 4$ SCFT with $c = 12$. This fact is proven as follows: In the RNS formalism supersymmetries that are gauged in space-time must arise from global currents on the worldsheet. This in particular means there must exist global space-time fermionic currents of the following form:

$$Q_+^A = \int dz e^{-\varphi/2} e^{iH/2} \Sigma^A \quad A = 1, 2, 3 \tag{A.1}$$

(A.1) is the standard FMS vertex which is the holomorphic part of the gravitino vertex operator at zero momentum. As usual the free fermions ψ^0 and ψ^1 corresponding to the flat two dimensional space-time have been bosonized into a chiral boson H and (A.1) is in the standard $(-\frac{1}{2})$ picture so Σ^A must have dimension $\frac{1}{2}$. Furthermore the space-time supersymmetry algebra without central charges is of the form:

$$\{Q_+^A, Q_+^B\} = \delta^{AB} P_+ \tag{A.2}$$

which requires Σ^A to satisfy the following OPEs:

$$\Sigma^A(z)\Sigma^B(w) = \delta^{AB} \frac{1}{(z-w)} \tag{A.3}$$

which automatically identify Σ^A as free Majorana fermions. One can choose a pair of fermions out of the three free Majorana fermions and bosonize the pair as:

$$\frac{1}{2}(\Sigma^1 + i\Sigma^2) = e^{i\phi} \tag{A.4}$$

This defines a U(1) current:

$$J_3 = 2i\partial\phi \tag{A.5}$$

which is actually an R-current. Using the remaining free Majorana fermion Σ^3 we can define two more U(1) generators:

$$J^\pm =: e^{\pm i\phi}\Sigma^3 : \tag{A.6}$$

(J^\pm, J_3) together generate the SU(2) Kac-Moody algebra at level $k = 2$. Therefore the internal SCFT turns out to have a small $\mathcal{N} = 4$ SCA with $c = 12$. This corresponds to the case of hyper-Kähler four-fold compactifications.⁷ In the large radius limit the worldsheet description of a hyper-Kähler four-fold compactification is via a $\mathcal{N} = 4$ supersymmetric sigma model which is also conformally invariant and leads to a SCFT with small $\mathcal{N} = 4$ SCA and $c = 12$. The moduli of the SCFT are the $\mathcal{N} = 4$ chiral primaries.

In order to deform the $\mathcal{N} = (4, 4)$ worldsheet SCFT one adds operators of the form:

$$\delta S = \int d^2z \mathcal{O}(z, \bar{z}) \tag{A.7}$$

To preserve conformal invariance \mathcal{O} must be a dimension (1, 1) operator. However in order to preserve $\mathcal{N} = (4, 4)$ worldsheet supersymmetry we require more. A $\mathcal{N} = (4, 4)$ SCA has four left-moving (and four right-moving) supercharges which can be denoted as G^\pm and \tilde{G}^\pm . The \pm indices indicate the U(1) R-charge of these operators under J_3 . In order to preserve $\mathcal{N} = (2, 2)$ supersymmetry generated by G^\pm and J_3 we require the operator \mathcal{O}

⁷There is also the possibility of getting six supercharges starting with two commuting $\mathcal{N} = 4$ SCFTs each with $c = 6$ for the left movers and having a $\mathcal{N} = 2$ SCA for the right-movers. This way the four supercharges from the left-moving sector and two from the right-moving sector yield in total six supercharges. This way however we end up with $\mathcal{N} = (4, 2)$ space-time supersymmetry from type IIA and $\mathcal{N} = (0, 6)$ supersymmetry from type IIB.

to be the top component of a chiral superfield whose bottom component is a chiral primary operator. That is, given a ϕ_i annihilated by $G_{-\frac{1}{2}}^+$ and carrying charge +1 and conformal dimension $\frac{1}{2}$ one deforms the action S into:

$$S' = S + \int d^2z (t^i G^- \phi_i + \bar{t}^i G^+ \bar{\phi}_i) \tag{A.8}$$

where by $G^+ \phi_i$ one means picking the z^{-1} pole of the $G^+ \phi_i$ OPE. Under what circumstances will a deformation of the form (A.8) respect $\mathcal{N} = 4$ superconformal invariance? For this the deformation (A.8) must be a SU(2) singlet. It is obviously a singlet under U(1) generated by J so we need to only check invariance under J^\pm . This requires:

$$\tilde{G}^+ \phi_i = 0 \tag{A.9}$$

That is ϕ_i is a $\mathcal{N} = 4$ primary with dimension $\frac{1}{2}$. This is the standard result that the CFT moduli arise from $\mathcal{N} = 4$ primary operators with dimension $\frac{1}{2}$.

B. Other theories with six supercharges

Let us explore the other possibilities that give us $\mathcal{N} = (3, 3)$ supersymmetry in 2d. As mentioned before there appears to be one possibility leading to $\mathcal{N} = (4, 2)$ space-time supersymmetry in 2d which we expect to be anomalous. What are the other possibilities? Equation (A.1) is not the only possibility. The other possibility is to obtain three supercharges from left movers as:

$$Q_+^A = \int dz e^{-\frac{1}{2}\varphi} e^{i\frac{H}{2}} \Sigma^A \quad A = 1, 2 \tag{B.1}$$

$$Q_+ = \int dz e^{-\frac{1}{2}\varphi} e^{-i\frac{H}{2}} \Sigma \tag{B.2}$$

which would give rise to $\mathcal{N} = (2, 1)$ supersymmetry from the left-movers. Together with the right movers one can obtain $\mathcal{N} = (3, 3)$ supersymmetry. Unfortunately, this will not work for the following reason. The OPEs following from (B.1) and (B.2) imply that Σ^A are free and can be bosonized as in (A.4). However there are non-trivial OPEs between the bosonized field ϕ and Σ of the form:

$$(e^{i\phi}(z) \pm e^{-i\phi(z)})\Sigma(w) = (z-w)^{\frac{1}{2}} \quad \Sigma(z)\Sigma(w) = \frac{1}{(z-w)} \tag{B.3}$$

In a standard fashion one can decouple ϕ from Σ by writing:

$$\Sigma = \sum_q : e^{i\frac{q}{2}\phi} \Sigma_q : \tag{B.4}$$

It is easy to see that (B.3) cannot hold for this form of Σ since there are always singularities of the form $z^{-\frac{q}{2}}$ and $z^{\frac{q}{2}}$ simultaneously, rather than the $z^{\frac{1}{2}}$ singularity alone that is expected in (B.3). In other words there is no way to make a supersymmetry algebra of the form (B.1) and (B.2). This leaves us with only very few possibilities for vacua with

$\mathcal{N} = 3$ supersymmetry in 2d. They appear upon turning on RR fluxes or orientifolding in compactifications with higher degree of supersymmetry.

For example we can contemplate whether starting from $\mathcal{N} = 4$ supersymmetric compactifications in 3d we can turning on flux that breaks $\mathcal{N} = 4$ to $\mathcal{N} = 3$. Unfortunately this cannot happen, since the only $\mathcal{N} = 4$ supersymmetric backgrounds are either of $K3 \times K3$ type or $T^2 \times CY_3$ in general. For $K3 \times K3$ it is only possible to break $\mathcal{N} = 4$ to $\mathcal{N} = 2$ by turning on fluxes, while $\chi(T^2 \times CY_3) = 0$ so no fluxes are allowed.

It is now known that the class of compactifications studied in [13] is not the most general supersymmetric solution with $\mathcal{N} = 2$ supersymmetry [24]. One way to interpret the solutions in [24] is in terms of calibration conditions for $M5$ -branes wrapping supersymmetric cycles. For example we can consider five-branes wrapping SLAG cycles of a CY 4-fold. Including the effect of backreaction we expect a $\mathcal{N} = 1$ supersymmetric vacuum in 3d, which is not a spin(7) vacuum. Rather it belongs to the class studied in [24]. This way, it is clear that the only way to generate $\mathcal{N} = 3$ supersymmetric vacua in \mathcal{M} -theory is to consider vacua with with $\mathcal{N} = 6$ supersymmetry in 3d, and turn on G -flux that breaks $\mathcal{N} = 6$ to $\mathcal{N} = 3$. There is no compact manifold which upon compactification \mathcal{M} -theory yields $\mathcal{N} = 6$ supersymmetry in 3d.

In fact, even [24] turns out to obtain a restricted class of \mathcal{M} -theory compactifications, and a more general class of compactifications on eight-manifolds is possible [22].⁸ It would be interesting to understand if in the case of $\mathcal{N} = 3$ supersymmetry the solutions to this more general class of compactifications allow us to obtain $\mathcal{N} = 3$ moduli spaces in 3d, of which the hyper-Kähler four-fold moduli space will be a strict subset.

It therefore appears that the only other way of obtaining $\mathcal{N} = 3$ vacua would be to start with even more symmetric spaces like tori and orbifolds/ orientifolds thereof and turn on supersymmetry breaking G -fluxes to end up with a $\mathcal{N} = 3$ supersymmetric vacuum in 3d. It would be nice to exhibit such examples, as they will be expected to provide insight into the HK moduli space.

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